STATISTICS IN TRANSITION new series, June 2026
Vol. 27, No. 2, pp. xx–XX, https://doi.org/10.59139/stattrans-2026-xxx
Received – xx.xx.xxxx; accepted – xx.xx.xxxx

Bayesian sensitivity of insurance premium in collective risk model under bivariate prior with dependent frequency and severity of claims

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Abstract

This study deals with the problem of robustness of the collective and Bayes premiums under uncertainty of prior knowledge. The inaccuracy of the prior knowledge concerns the disturbance of independence between variables describing the frequency and average value of claims. Traditionally, these variables are independent, but in applications it is not always the case. Two classes of priors are presented: in the first class, the FGM copula is applied, while in the second one, the dependence between two contaminated priors is shown. In both classes, priors have the form of a linear combination of known bivariate probability distributions. The ranges of collective and Bayes premiums are calculated and prior and posterior regret gamma-minimax premiums are presented as the optimal premiums. Despite the very mild or small dependence, its influence on the premiums, especially on the bonus-malus factor, is relatively significant.

Key words: classes of priors, FGM copula, ε-contamination, posterior regret Γ-minimax premium, mean square error, bonus-malus factor.

1. Introduction

One of the most important aims in insurance is to estimate the premium. The premium is defined as a functional H which assigns to a real risk S a non-negative real number. In this paper a collective risk model is presented and the risk variable has a probability distribution function (p.d.f.) depending on a bivariate unknown parameter (λ,θ). The first parameter describes the expected value of the number of claims in one period (in the paper, one year), and the second one describes the expected value of the severity of a claim. There are many principles according to which premiums are calculated. Mamy of them are presented in Heilmann (1989), Gómez-Déniz et al. (1999), Boratyńska (2008), Furman and Zitikis (2008), Young (2004). We consider the net premium in the form $H(S)=H(\lambda,\theta)=a\lambda\theta$, where $a>0$ is known. It is called the individual premium. The parameters λ and θ are unknown, so we ought to estimate H.

We will use the Bayesian methodology to combine the prior knowledge about parameters (defined by a prior distribution) with the knowledge about the history of the risk in the form of a random sample, where the probability distribution of this random variable depends on the parameters. The quality of an estimator is measured by the expected value of

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a squared error loss function. Thus having some prior information about parameters, described by a prior distribution π (we will use the same notation for a probability distribution and its density (p.d.f.) with respect to the chosen measure on a probability space), and minimizing the expected squared error loss we obtain the collective premium $H^C(\pi)$ equal to the expected value of $H(\lambda,\theta)$ under the prior π. This premium is a premium in a class of risk, because the prior expresses the population behaviour of an unknown parameter. Introducing a random sample x with a p.d.f. dependent on the parameter (λ,θ) and minimizing the expected value of the squared error loss if the parameter has the posterior distribution, we calculate a Bayes premium $H^B(\pi,x)$ equal to the expected value of $H(\lambda,\theta)$ under the posterior distribution $\pi(\cdot|x)$. This premium combines knowledge about the population and about one considered risk (a policy).

The collective and Bayes premiums depend on a choice of a prior. The elicitation of a prior is difficult and can be uncertain. To model uncertainty of the prior information the robust Bayesian inference uses a class Γ of priors. In the literature, there are many different classes Γ of priors. For general references, see Berger (1994), Ríos Insua and Ruggeri (2000), Ruggeri et al. (2021). In insurance, robust Bayesian analysis has been considered in many papers, for example: Gómez-Déniz et al. (2002), Gómez-Déniz (2009), Peters et al. (2017), Sánchez-Sánchez et al. (2019), Boratyńska (2021, 2022), Ruggeri et al. (2025). For some recent papers applying robustness of Bayesian procedures in different fields see Tomer and Rai (2021), Ho (2023), Harrouche et al. (2025).

The main objective of this study is to examine how both collective and Bayes premiums respond to some uncertainty related to the independence between variables describing the frequency and the average severity of claims. We consider two classes of priors. The priors have the form of a linear combination of fixed priors. In the first class Γ_1, the Farlie-Gumbel-Morgenstern (FGM) copula with the fixed marginals is considered (for definition and properties of FGM see Nelsen (2006)). In the second class Γ_2, the marginal priors for both parameters are the mixtures of two fixed priors (they have the form of ε-contaminated priors, widely considered in the literature about robustness), but the variables have some degree of dependence.

Having a class of priors we have a set of collective premiums and a set of Bayes premiums. The sensitivity of the considered premium is measured by the range of the set, when priors run over the class Γ. If the range is small (according to the statisticians' or experts' judgements), then any prior can be chosen since all of them lead to similar results (see Berger (1994), Ríos Insua and Ruggeri (2000), among others). On the other hand, the practitioner faces a problem of choosing the optimal estimator. There are several concepts of optimal rules, for details see Ríos Insua and Ruggeri (2000), Hu and Xiao (2021), and references therein. As an optimal procedure we consider the prior and posterior regret Γ-minimax estimator (PRGM estimator). The selected optimal rules provide the estimators, which minimize the largest possible increase in risk resulting from making the wrong choice of a prior distribution. Their values depend on the bands of a set of the considered premiums calculated with respect to the priors belonging to the class Γ. Thus, computing a PRGM estimator is relatively simple.

In calculation of the insurance premium the number and severity of claims are typically assumed to be independent, but it is sometimes more reasonable to allow some dependence (see Lemaire (1995), Gschlößl and Czado (2007), Shi et al. (2015), Lee et al. (2019), Lee and Shi (2019), Oh et al. (2020)). There are different tools to model dependence, for exampe: regression models, generalized linear models, copulas, but there are not many papers where the dependence is modelled using appropriate priors. For references, see Cheung et al. (2021). Some dependence between parameters λ and θ have been considered by Hernandez-Bastida et al. (2009), Cheung et al. (2021), Gomez-Deniz (2016), Ruggeri et al. (2025), among others, but without PRGM procedures.

The paper is organized as follows. In Section 2 the base model is presented and the measures of robustness and optimal premiums are defined. The classes of priors and the main results are in Section 3. In Section 4 the results are applied to a numerical example. Additionally, we illustrate the influence of uncertainty of the prior on the bonus-malus coefficient. Finally, in Section 5 we provide concluding remarks.

2. Bayesian collective risk model and measures of robustness

For a given contract (risk) let N_h be a random variable describing the number of claims in a year h, and $Y_{h1}, Y_{h2}, \ldots$ be random variables describing the severity of claims, and let $S_h = \sum_{i=1}^{N_h} Y_{hi}$ ($S_h = 0$ if $N_h = 0$) be an aggregate claim amount. Let λ and θ be two positive continuous random variables. We assume that given λ, the random variable N_h has the Poisson distribution $Poiss(\lambda)$, and given θ, random variables Y_{hi}, $i = 1, 2, \ldots$, are i.i.d. with the gamma distribution $Gamma(a; \frac{1}{\theta})$, where $a > 0$ is known, and gamma distribution $Gamma(\alpha; \beta)$ with parameters $\alpha, \beta > 0$ has the p.d.f. equal to

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$$

for $x > 0$. Given λ and θ, random variables N_h, Y_{hi}, $i = 1, 2, \ldots$, are independent.

Assume that λ has the prior distribution $\pi_{10} = Gamma(\mu_{10}k_{10}; k_{10})$ and θ has the prior distribution $\pi_{20} = IGamma(k_{20} + 1; \mu_{20}k_{20})$, where k_{i0}, μ_{i0}, for $i = 1, 2$, are fixed positive numbers, $k_{20} > 1$, and $IGamma(\gamma + 1; \delta)$ is an inverse gamma distribution with parameters $\gamma > 1$ and $\delta > 0$ and the p.d.f. equal to

$$\pi(\theta|\gamma, \delta) = \frac{\delta^{\gamma+1}}{\Gamma(\gamma+1)} \theta^{-\gamma-2} \exp\left(-\frac{\delta}{\theta}\right)$$

for $\theta > 0$. Suppose that λ and θ are independent. Thus, the bivariate random variable (λ, θ) has the prior p.d.f. $\pi_{00}(\lambda, \theta) = \pi_{10}(\lambda)\pi_{20}(\theta)$ with respect to the Lebesgue measure on the space $\Lambda \times \Theta = (0, +\infty) \times (0, +\infty)$. The parametrization of the prior gamma and inverse gamma distributions is chosen so that μ_{10} and μ_{20} (similarly, later in the paper, μ_{1i} and μ_{2j}) correspond to the prior expected values of the variables λ and θ. From an actuary's perspective, these parameters are relatively easy to estimate. Furthermore, in the definitions of the prior and posterior distributions, as well as in the parameters of these distributions throughout the paper, the subscript $1i$ indicates that the distribution refers to the variable λ, while the subscript $2j$ refers to the variable θ. The symbol μ with the corresponding indices always denotes the prior expected value of the respective distribution.

The p.d.f. π_{00} is the conjugate prior. Hence, given the data

$$(N^t, S^t) = (N_1, S_1, \dots, N_t, S_t)$$

(the history of the contract for t years), the posterior distribution $\pi_{00}(\cdot|N^t, S^t)$ of (λ, θ) is a product of $Gamma(N + \mu_{10}k_{10}; t + k_{10})$ and $IGamma(Na + k_{20} + 1; S + k_{20}\mu_{20})$ distributions, where $N = \sum_{h=1}^{t} N_h$ and $S = \sum_{h=1}^{t} S_h$. Thus, given (N^t, S^t), r.vs. λ and θ are independent.

In this model the net individual premium is equal to $H(\lambda, \theta) = E(S_h|\lambda, \theta) = a\lambda\theta$. Under the squared error loss function the collective premium and the Bayes premium are equal to

$$H_0^C = H^C(\pi_{00}) = E_{\pi_{00}}(a\lambda\theta) = a\mu_{10}\mu_{20}, \tag{1}$$

$$H_0^B = H^B(\pi_{00}, N^t, S^t) = E_{\pi_{00}}(a\lambda\theta|N^t, S^t) = a\frac{N + \mu_{10}k_{10}}{t + k_{10}} \frac{S + k_{20}\mu_{20}}{Na + k_{20}}. \tag{2}$$

Assume that the prior knowledge is not enough to elicit one prior distribution. Therefore, consider a class Γ of priors on the space $\Lambda \times \Theta$, and let $H^C(\pi)$, $H^B(\pi, N^t, S^t)$ be the collective and the Bayes premium under the square error loss function and a prior $\pi \in \Gamma$. Then we consider the oscillations:

$$r(H^C, \Gamma) = \sup_{\pi \in \Gamma} H^C(\pi) - \inf_{\pi \in \Gamma} H^C(\pi),$$

$$r(H^B(\cdot, N^t, S^t), \Gamma) = \sup_{\pi \in \Gamma} H^B(\pi, N^t, S^t) - \inf_{\pi \in \Gamma} H^B(\pi, N^t, S^t),$$

as measures of robustness for the collective and the Bayes premiums. Besides the measure of range of the collective and Bayes premiums we would like to choose an optimal procedure and we decided on the regret gamma-minimax estimation. The prior and the posterior regret under the squared error loss of a decision d are respectively equal to

$$reg(\pi, d) = E_\pi(a\lambda\theta - d)^2 - E_\pi(a\lambda\theta - H^C(\pi))^2$$

and

$$reg(\pi, d|N^t, S^t) = E_\pi\left((a\lambda\theta - d)^2|N^t, S^t\right) - E_\pi\left((a\lambda\theta - H^B(\pi, N^t, S^t))^2|N^t, S^t\right).$$

They measure the loss of optimality of the risk if we choose an estimate d instead of the best estimate (in our problem collective and Bayes premiums), which minimizes the expected loss, prior and posterior, respectively. Now, the prior and posterior regret gamma-minimax premiums: $H_{PR}^C(\Gamma)$ and $H_{PR}^B(\Gamma, N^t, S^t)$, satisfy conditions:

$$\sup_{\pi \in \Gamma} reg(\pi, H_{PR}^C(\Gamma)) = \inf_d \sup_{\pi \in \Gamma} reg(\pi, d),$$

$$\sup_{\pi \in \Gamma} reg(\pi, H_{PR}^B(\Gamma, N^t, S^t)|N^t, S^t) = \inf_d \sup_{\pi \in \Gamma} reg(\pi, d|N^t, S^t).$$

This methodology is based on the idea that the optimal action minimizes the supremum of the function over distributions in the class Γ. The prior and posterior regret gamma-minimax premiums are equal to:

$$H^C_{PR}(\Gamma) = \frac{1}{2}\left(\sup_{\pi\in\Gamma} H^C(\pi) + \inf_{\pi\in\Gamma} H^C(\pi)\right), \quad (3)$$

$$H^B_{PR}(\Gamma, N^t, S^t) = \frac{1}{2}\left(\sup_{\pi\in\Gamma} H^B(\pi, N^t, S^t) + \inf_{\pi\in\Gamma} H^B(\pi, N^t, S^t)\right) \quad (4)$$

(for details see Ríos Insua et al. (1995)).

3. Robustness of the collective and Bayes premiums and PRGM premiums

Two classes of priors are considered. The prior distributions belonging to the classes are linear combinations of the known distributions. Their p.d.fs have the form

$$\pi(\lambda, \theta) = \sum_{i=1}^{r}\sum_{j=1}^{s} \alpha_{ij}\pi_{1i}(\lambda)\pi_{2j}(\theta),$$

where $\pi_{1i} = Gamma(\alpha_i; \beta_i)$ and $\pi_{2j} = IGamma(\gamma_j + 1; \delta_j)$, where α_i, β_i, γ_j, δ_j are positive numbers, $\gamma_j > 1$, and α_{ij} are fixed numbers, such that π is a valid bivariate p.d.f. The following lemma will be useful to calculate posterior distributions and premiums.

Lemma 1. (see also Cheung et al. (2021)) *If* $\pi(\lambda, \theta) = \sum_{i=1}^{r}\sum_{j=1}^{s} \alpha_{ij}\pi_{1i}(\lambda)\pi_{2j}(\theta)$ *then, given* (N^t, S^t)*, the posterior distribution is equal to*

$$\pi(\lambda, \theta|N^t, S^t) = \frac{1}{A}\sum_{i=1}^{r}\sum_{j=1}^{s} \alpha^*_{ij}\pi_{1i}(\lambda|N^t, S^t)\pi_{2j}(\theta|N^t, S^t), \quad (5)$$

where the posterior p.d.f. $\pi_{1i}(\lambda|N^t, S^t)$ *is the p.d.f. of the distribution* $Gamma(N + \alpha_i; t + \beta_i)$ *and* $\pi_{2j}(\theta|N^t, S^t)$ *is the p.d.f. of the distribution* $IGamma(Na + \gamma_j + 1; S + \delta_j)$*, and*

$$\alpha^*_{ij} = \alpha_{ij}\frac{\beta_i^{\alpha_i}\Gamma(N+\alpha_i)}{\Gamma(\alpha_i)(\beta_i + t)^{\alpha_i+N}} \cdot \frac{\delta_j^{\gamma_j+1}\Gamma(aN+\gamma_j+1)}{(S+\delta_j)^{\gamma_j+aN+1}\Gamma(\gamma_j+1)}, \quad A = \sum_{i=1}^{r}\sum_{j=1}^{s}\alpha^*_{ij}. \quad (6)$$

Under the prior π *the collective and Bayes premiums are equal to*

$$H^C(\pi) = aE_\pi(\lambda\theta) = a\sum_{i=1}^{r}\sum_{j=1}^{s}\alpha_{ij}\frac{\alpha_i\delta_j}{\beta_i\gamma_j},$$

$$H^B(\pi, N^t, S^t) = a\sum_{i=1}^{r}\sum_{j=1}^{s}\frac{\alpha^*_{ij}}{A}\frac{(N+\alpha_i)(S+\delta_j)}{(t+\beta_i)(Na+\gamma_j)}. \quad \blacksquare$$

3.1. The first class

Consider the FGM copula defined by $C(u,v) = uv + \omega uv(1-u)(1-v)$, for $u, v \in (0,1)$, $\omega \in [-1,1]$, and its derivative

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v} = 1 + \omega(1-2u)(1-2v).$$

Let $k_{10}\mu_{10}$ and $k_{20} > 1$ be positive integers. Consider the following class of priors:

$$\Gamma_1 = \{\pi_\omega : \ \pi_\omega(\lambda,\theta) = c\left(\Pi_{10}(\lambda), \Pi_{20}(\theta)\right)\pi_{10}(\lambda)\pi_{20}(\theta) : \omega \in [\omega_1, \omega_2]\},$$

where $\Pi = 1 - \bar{\Pi}$ is the cumulative distribution function (c.d.f.) for the p.d.f. π and $\omega_1 < \omega_2$ and $\omega_1, \omega_2 \in [-1, \ 1]$ are fixed.

The priors belonging to the class Γ_1 have fixed marginal priors equal to π_{10} and π_{20} and $\pi_{\omega=0}(\lambda,\theta) = \pi_{00}(\lambda,\theta) = \pi_{10}(\lambda)\pi_{20}(\theta)$. The parameter ω is the dependence parameter and the Kendall and Spearman coefficients are equal to $\tau(\omega,\lambda,\theta) = \frac{2\omega}{9}$ and $\rho(\omega,\lambda,\theta) = \frac{\omega}{3}$ (see Nelsen (2006)). Hence, the mild dependence between λ and θ is assumed.

Lemma 2. *If π_{10} is $Gamma(\mu_{10}k_{10}; k_{10})$ and π_{20} is $IGamma(k_{20}+1; \mu_{20}k_{20})$, where $k_{10}\mu_{10}$ and k_{20} are positive integers, then the p.d.f. π_ω is equal to*

$$\pi_\omega(\lambda,\theta) = \pi_{10}(\lambda)\pi_{20}(\theta) + \omega \sum_{i=0}^{l_1}\sum_{j=0}^{l_2} \alpha_{ij}\pi^*_{1i}(\lambda)\pi^*_{2j}(\theta), \tag{7}$$

where $l_1 = k_{10}\mu_{10}$, $l_2 = k_{20}+1$, $\alpha_{00} = -1$, $\pi^*_{10} = \pi_{10}$, $\pi^*_{20} = \pi_{20}$,

$$\pi^*_{1i} = Gamma(i + l_1 - 1; 2k_{10}), \quad \pi^*_{2j} = IGamma(j + l_2 - 1; 2k_{20}\mu_{20}),$$

$$\alpha_{i0} = \frac{2}{2^{i+l_1-1}}\binom{i+l_1-2}{i-1}, \quad \alpha_{0j} = \frac{2}{2^{j+l_2-1}}\binom{j+l_2-2}{j-1}, \quad \alpha_{ij} = -\alpha_{i0}\alpha_{0j},$$

for $i = 1,\ldots,l_1$, $j = 1,\ldots,l_2$.

Proof. Denote $l_1 = k_{10}\mu_{10}$ and $l_2 = k_{20}+1$. Similarly to the article Cheung et al. (2021), we present the survival function of π_{10} and the c.d.f. of π_{20} for $\lambda, \ \theta > 0$ as follows:

$$\bar{\Pi}_{10}(\lambda) = \int_\lambda^{+\infty} \frac{k_{10}^{l_1}}{(l_1-1)!} u^{l_1-1}\exp\left(-k_{10}u\right)du = \sum_{n=0}^{l_1-1}\frac{k_{10}^n}{n!}\lambda^n \exp\left(-k_{10}\lambda\right),$$

$$\Pi_{20}(\theta) = \int_0^\theta \frac{(k_{20}\mu_{20})^{l_2}}{(l_2-1)!u^{l_2+1}}\exp\left(\frac{-k_{20}\mu_{20}}{u}\right)du$$
$$= \sum_{n=0}^{l_2-1}\frac{(k_{20}\mu_{20})^n}{n!}\theta^{-n}\exp\left(\frac{-k_{20}\mu_{20}}{\theta}\right).$$

The p.d.f π_ω is equal to

$$\pi_\omega(\lambda,\theta) = \left(1 + \omega(2\bar{\Pi}_{10}(\lambda)-1)(1-2\Pi_{20}(\theta))\right)\pi_{10}(\lambda)\pi_{20}(\theta)$$

$$= \left(1+\omega\left(-1+2\bar{\Pi}_{10}(\lambda)+2\Pi_{20}(\theta)-4\bar{\Pi}_{10}(\lambda)\Pi_{20}(\theta)\right)\right)\pi_{10}(\lambda)\pi_{20}(\theta).$$

By substituting $\bar{\Pi}_{10}(\lambda)$ and $\Pi_{20}(\theta)$ we obtain the assertion. ■

All prior distributions from the class Γ_1 have the form given in Lemma 1. Applying Lemma 1 and Lemma 2 we have

$$E_{\pi_\omega}(\lambda\theta) = \mu_{10}\mu_{20} + \omega\sum_{i=0}^{l_1}\sum_{j=0}^{l_2}\alpha_{ij}\mu_{1i}^*\mu_{2j}^*,$$

$$\mu_{10}^* = \mu_{10}, \quad \mu_{1i}^* = \frac{i+k_{10}\mu_{10}-1}{2k_{10}}, \quad \mu_{20}^* = \mu_{20}, \quad \mu_{2j}^* = \frac{2k_{20}\mu_{20}}{j+k_{20}-1},$$

for $i=1,\ldots,l_1$ and $j=1,\ldots,l_2$. Thus, assuming $l_2>2$, the Pearson correlation coefficient is equal to

$$Corr_{\pi_\omega}(\lambda,\theta) = \omega\sum_{i=0}^{l_1}\sum_{j=0}^{l_2}\alpha_{ij}\mu_{1i}^*\mu_{2j}^*\frac{\sqrt{l_1(l_2-2)}}{\mu_{20}\mu_{10}} \tag{8}$$

and the collective premium and $Corr_{\pi_\omega}(\lambda,\theta)$ are the linear monotone functions of ω.

Given (N^t,S^t) and applying (7), (5), (6), we obtain the posterior p.d.f. and the Bayes premium

$$\pi_\omega(\lambda,\theta|N^t,S^t)$$
$$= \frac{-\alpha_{00}^*}{A^*}\pi_{10}(\lambda|N^t,S^t)\pi_{20}(\theta|N^t,S^t) + \omega\sum_{i=0}^{l_1}\sum_{j=0}^{l_2}\frac{\alpha_{ij}^*}{A^*}\pi_{1i}^*(\lambda|N^t,S^t)\pi_{2j}^*(\theta|N^t,S^t),$$

$$H^B(\pi_\omega,N^t,S^t)) = aE_{\pi_\omega}(\lambda\theta|N^t,S^t), \tag{9}$$

where the posterior p.d.fs. $\pi_{1i}^*(\lambda|N^t,S^t)$ and $\pi_{2j}^*(\theta|N^t,S^t)$ are the p.d.fs. of the distributions $Gamma(N+i+l_1-1;t+2k_{10})$ and $IGamma(Na+j+l_2-1;S+2k_{20}\mu_{20})$, for $i=1,\ldots,l_1$ and $j=1,\ldots,l_2$, respectively, and

$$\pi_{10}^*(\lambda|N^t,S^t) = \pi_{10}(\lambda|N^t,S^t), \quad \pi_{20}^*(\theta|N^t,S^t) = \pi_{20}(\theta|N^t,S^t),$$

$$\alpha_{00}^* = \frac{-k_{10}^{l_1}\Gamma(N+l_1)}{\Gamma(l_1)(k_{10}+t)^{l_1+N}}\cdot\frac{(\mu_{20}k_{20})^{l_2}\Gamma(aN+l_2)}{(S+\mu_{20}k_{20})^{l_2+aN}\Gamma(l_2)},$$

$$\alpha_{i0}^* = \frac{2}{(i-1)!}\frac{k_{10}^{i+l_1-1}\Gamma(N+i+l_1-1)}{\Gamma(l_1)(2k_{10}+t)^{N+i+l_1-1}}\cdot\frac{(\mu_{20}k_{20})^{l_2}\Gamma(aN+l_2)}{(S+\mu_{20}k_{20})^{l_2+aN}\Gamma(l_2)},$$

$$\alpha_{0j}^* = \frac{2}{(j-1)!}\frac{k_{10}^{l_1}\Gamma(N+l_1)}{\Gamma(l_1)(k_{10}+t)^{l_1+N}}\cdot\frac{(\mu_{20}k_{20})^{j+l_2-1}\Gamma(aN+j+l_2-1)}{(S+2\mu_{20}k_{20})^{j+l_2+aN-1}\Gamma(l_2)},$$

$$\alpha_{ij}^* = \frac{\alpha_{i0}^*\alpha_{0j}^*}{\alpha_{00}^*}, \quad A^* = -\alpha_{00}^* + \omega\sum_{i=0}^{l_1}\sum_{j=0}^{l_2}\alpha_{ij}^*, \tag{10}$$

for $i=1,\ldots,l_1,\ j=1,\ldots,l_2$. The Bayes premium is a homographic function of ω (see (9) and (10)), therefore, the supremum and infimum are achieved at the boundary of the interval $[\omega_1,\omega_2]$. Hence, applying (3) and (4), we obtain the theorem.

Theorem 1. *If the class of priors is equal to* Γ_1 *then the oscillations of the collective premium and the Bayes premium are equal to*

$$r(H^C,\Gamma_1) = a(\omega_2-\omega_1)\left|\sum_{i=0}^{l_1}\sum_{j=0}^{l_2}\alpha_{ij}\mu_{1i}^*\mu_{2j}^*\right|,$$

$$r(H^B(\cdot,N^t,S^t),\Gamma_1) = \left|H^B(\pi_{\omega=\omega_1},N^t,S^t) - H^B(\pi_{\omega=\omega_2},N^t,S^t)\right|.$$

The prior regret Γ*-minimax and the posterior regret* Γ*-minimax premiums are equal to*

$$H_{PR}^C(\Gamma_1) = a\mu_{10}\mu_{20} + \frac{\omega_1+\omega_2}{2}\sum_{i=0}^{l_1}\sum_{j=0}^{l_2}a\alpha_{ij}\mu_{1i}^*\mu_{2j}^*,$$

$$H_{PR}^B(\Gamma_1,N^t,S^t) = \frac{1}{2}\left(H^B(\pi_{\omega=\omega_1},N^t,S^t) + H^B(\pi_{\omega=\omega_2},N^t,S^t)\right). \qquad \blacksquare$$

Note that if $\omega_1 = -\omega_2$ then $H_{PR}^C(\Gamma_1)$ is equal to the collective premium for the basic prior (the prior, where λ and θ are independent). The relative range equal to $\frac{r(H^C,\Gamma_1)}{H_0^C}$ does not depend on μ_{20} and μ_{10} as long as $k_{10}\mu_{10}$ and k_{20} are fixed.

3.2. The second class

Assume two contaminated priors:

$$\pi^\varepsilon(\lambda) = (1-\varepsilon)\pi_{10}(\lambda) + \varepsilon\pi_{11}(\lambda), \quad \pi^\eta(\theta) = (1-\eta)\pi_{20}(\theta) + \eta\pi_{21}(\theta),$$

where $\pi_{1i} = Gamma(k_{1i}\mu_{1i};k_{1i})$ and $\pi_{2i} = IGamma(k_{2i}+1;k_{2i}\mu_{2i})$, for $i=0,1$, and $\varepsilon,\eta \in (0,1/2]$, $k_{1i}>0, k_{2i}>1$, $\mu_{1i},\mu_{2i}>0$ are fixed numbers. The elicited priors for both variables are a mixture of two probability distributions. Then the product of the measures on the product space $\Lambda\times\Theta$ has the form given in Lemma 1, namely

$$\pi^{\varepsilon,\eta}(\lambda,\theta) = (1-\varepsilon)(1-\eta)\pi_{10}(\lambda)\pi_{20}(\theta)$$

$$+\varepsilon(1-\eta)\pi_{11}(\lambda)\pi_{20}(\theta) + (1-\varepsilon)\eta\pi_{10}(\lambda)\pi_{21}(\theta) + \varepsilon\eta\pi_{11}(\lambda)\pi_{21}(\theta),$$

and the random variables λ and θ are independent. Now, consider the class

$$\Gamma_2 = \left\{\pi_\tau:\ \pi_\tau(\lambda,\theta) = \sum_{i=0}^{1}\sum_{j=0}^{1}\alpha_{ij}(\tau)\pi_{1i}(\lambda)\pi_{2j}(\theta),\ \tau\in[0,\min\{\varepsilon,\eta\}]\right\},$$

where $\alpha_{00}(\tau) = 1-\varepsilon-\eta+\tau,\ \alpha_{01}(\tau) = \eta-\tau,\ \alpha_{10}(\tau) = \varepsilon-\tau,\ \alpha_{11}(\tau) = \tau.$

The class Γ_2 contains the priors with the marginals equal to π^ε and π^η, but the variables λ and θ can be dependent. The independence is achieved if and only if $\tau = \varepsilon\eta$.

If the prior is π_τ then

$$E_{\pi_\tau}(\lambda\theta) =$$

$$\tau(\mu_{11}-\mu_{10})(\mu_{21}-\mu_{20}) + \varepsilon\mu_{20}(\mu_{11}-\mu_{10}) + \eta\mu_{10}(\mu_{21}-\mu_{20}) + \mu_{10}\mu_{20} \qquad (11)$$

and the covariance $Cov_{\pi_\tau}(\lambda,\theta)$ is equal to

$$Cov_{\pi_\tau}(\lambda,\theta) = (\tau-\varepsilon\eta)(\mu_{11}-\mu_{10})(\mu_{21}-\mu_{20}). \tag{12}$$

Hence, $Cov_{\pi_\tau}(\lambda,\theta)=0$ if $\tau=\varepsilon\eta$ (λ and θ are independent) or $\mu_{11}=\mu_{10}$ or $\mu_{21}=\mu_{20}$. Therefore, in the class Γ_2, there can be distributions describing the dependent variables λ and θ, but with $Cov(\lambda,\theta)=0$. For $(\mu_{11}-\mu_{10})(\mu_{21}-\mu_{20})>(<)0$ the covariance is a increasing (decreasing) function of τ and is negative (positive) for $\tau<\varepsilon\eta$. The behavior of the collective premium is presented in the following theorem.

Theorem 2. *If Γ_2 is the class of priors, then the oscillation of the collective premium is equal to*

$$r(H^C,\Gamma_2) = a\min\{\varepsilon,\eta\}|(\mu_{11}-\mu_{10})(\mu_{21}-\mu_{20})|,$$

and the optimal collective premium is

$$H^C_{PR}(\Gamma_2) = H^C\left(\pi_{\tau=\frac{\min\{\varepsilon,\eta\}}{2}}\right).$$

If $(\mu_{11}-\mu_{10})(\mu_{21}-\mu_{20})=0$ then $H^C(\pi_\tau)=H^C(\pi^{\varepsilon,\eta})$ does not depend on τ.

Proof. The collective premium under the prior π_τ is equal to $H^C(\pi_\tau)=aE_{\pi_\tau}(\lambda\theta)$, and it is a linear function of $\tau\in[0,\min\{\varepsilon,\eta\}]$ with the slope $(\mu_{11}-\mu_{10})(\mu_{21}-\mu_{20})$ (see (11)). Hence, applying the properties of a linear function, we obtain $H^C_{PR}(\Gamma_2)$ and $r(H^C,\Gamma_2)$. ■

Note that $H^C_{PR}(\Gamma_2)$ is equal to the collective premium for independent λ and θ if $\varepsilon=\frac{1}{2}$ or $\eta=\frac{1}{2}$, but regardless of the values of ε and η, $H^C_{PR}(\Gamma_2)$ is a collective premium under a certain prior belonging to Γ_2. For every $\varepsilon,\eta<\frac{1}{2}$ we have $\frac{1}{2}\min\{\varepsilon,\eta\}>\varepsilon\eta$. Thus, if $(\mu_{11}-\mu_{10})(\mu_{21}-\mu_{20})>(<)0$, then premium $H^C_{PR}(\Gamma_2)$ is greater (less) than $H^C(\pi_{\varepsilon\eta})$. If $\mu_{10}=\mu_{11}$ and $\mu_{20}=\mu_{21}$, then, comparing (1) and (11), $H^C(\pi_\tau)=H^C_0$ for every $\tau\in[0,\min\{\varepsilon,\eta\}]$.

Given the data (N^t,S^t) and the prior π_τ and applying Lemma 1 (see (5) and (6)) we obtain the posterior p.d.f.

$$\pi_\tau(\lambda,\theta|N^t,S^t) = \sum_{i=0}^{1}\sum_{j=0}^{1}\frac{\alpha^*_{ij}(\tau)}{A(\tau)}\pi_{1i}(\lambda|N^t,S^t)\pi_{2j}(\theta|N^t,S^t),$$

where $\pi_{1i}(\lambda|N^t,S^t)$ is the p.d.f. of the distribution $Gamma(N+\mu_{1i}k_{1i};t+k_{1i})$ and $\pi_{2j}(\theta|N^t,S^t)$ is the p.d.f. of the distribution $IGamma(Na+k_{2j}+1;S+k_{2j}\mu_{2j})$, and

$$\alpha^*_{ij}(\tau) = \alpha_{ij}(\tau)\frac{k_{1i}^{\mu_{1i}k_{1i}}\Gamma(N+\mu_{1i}k_{1i})}{\Gamma(\mu_{1i}k_{1i})(k_{1i}+t)^{\mu_{1i}k_{1i}+N}}\cdot\frac{(\mu_{2j}k_{2j})^{k_{2j}+1}\Gamma(aN+k_{2j}+1)}{(S+\mu_{2j}k_{2j})^{k_{2j}+aN+1}\Gamma(k_{2j}+1)},$$

and $A(\tau)=\sum_{i=0}^{1}\sum_{j=0}^{1}\alpha^*_{ij}(\tau)$. Hence, the Bayes premium is equal to

$$H^B(\pi_\tau,N^t,S^t) = a\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{\alpha^*_{ij}(\tau)}{A(\tau)}\frac{N+\mu_{1i}k_{1i}}{t+k_{1i}}\frac{S+k_{2j}\mu_{2j}}{Na+k_{2j}},$$

and it is a homographic function of the variable τ, therefore the supremum and infimum are achieved at the boundary of the interval $[0, \min\{\varepsilon, \eta\}]$. Hence, we obtain the following theorem.

Theorem 3. *Assuming Γ_2 class of priors and the history of claim number and claim amount in the past t years N^t, S^t, the range of the Bayes premium is equal to*

$$r(H^B(\cdot, N^t, S^t), \Gamma_2) = |H^B(\pi_{\tau=\min\{\varepsilon,\eta\}}, N^t, S^t) - H^B(\pi_{\tau=0}, N^t, S^t)|$$

and the posterior regret Γ-minimax premium is equal to

$$H^B_{PR}(\Gamma_2, N^t, S^t) = \frac{1}{2}\left(H^B(\pi_{\tau=\min\{\varepsilon,\eta\}}, N^t, S^t) + H^B(\pi_{\tau=0}, N^t, S^t)\right). \qquad \blacksquare$$

Note that regardless of the value of the product $(\mu_{11} - \mu_{10})(\mu_{21} - \mu_{20})$, the Bayes premium depends on the value of the parameter τ.

4. Numerical example – sensitivity of bonus-malus system

The bonus-malus system (BM) adjusts the insurance premium based on the policyholder's claims history. With a good history (no claims), it lowers the premium, while with claims, it increases it. The BM coefficient takes into account the ratio of the Bayes premium (dependent on claims history) to the collective premium (considered as the base premium). The BM coefficient shows what percentage of the basic collective premium is represented by the Bayes premium for a given claims scenario. It often considers the number of claims. In the following example, besides the range of Bayes and collective premiums and PRGM premiums, we will consider the BM coefficient based on the number and severity of claims. We define the BM coefficient in the basic model as

$$BM(\pi_{00}, N^t, S^t) = \frac{H^B(\pi_{00}, N^t, S^t)}{H^C(\pi_{00})}.$$

When the prior distribution belongs to a class Γ of prior distributions, we consider the coefficients:

$$BM_{min} = \inf_{\pi\in\Gamma} \frac{H^B(\pi, N^t, S^t)}{H^C(\pi_{00})} \quad \text{and} \quad BM_{max} = \sup_{\pi\in\Gamma} \frac{H^B(\pi, N^t, S^t)}{H^C(\pi_{00})}$$

and compare them with the value

$$BM_{PR}(\Gamma, N^t, S^t) = \frac{H^B_{PR}(\Gamma, N^t, S^t)}{H^C_{PR}(\Gamma)},$$

which corresponds to the situation when, for the considered class of prior distributions, we use the prior and posterior regret gamma-minimax premiums.

Consider two base models. In the first one (M1), assume $k_{10} = 2.5$ and $\mu_{10} = 0.4$, in the second model (M2), $k_{10} = \mu_{10} = 1$. Thus, we assume greater expected frequency in the model M2 (the similar prior was considered in Rugger et al. (2025)). In both models $k_{20} = 2$ and $\mu_{20} = 200$. Consider the class Γ_1 with $\omega_1 = -\omega_2 = -1$. The behavior of the collective premium is presented in Table 1. The last two columns show the relative sensitivity of H^C, namely

$$R_{min} = \frac{\inf_{\pi\in\Gamma_1} H^C(\pi)}{H^C(\pi_{00})}, \qquad R_{max} = \frac{\sup_{\pi\in\Gamma_1} H^C(\pi)}{H^C(\pi_{00})}.$$

The collective premium is sensitive to the assumption of independence between frequency and severity of claims. In the class Γ_1 the Pearson correlation (8) is in the interval $[-0.1875, 0.1875]$, the Kendall and Spearman coefficients are in intervals $[-\frac{2}{9}, \frac{2}{9}]$ and $[-\frac{1}{3}, \frac{1}{3}]$ respectively, but the collective premium is in the interval $[0.81H^C(\pi_{00}), 1.19H^C(\pi_{00})]$.

Table 1. Sensitivity of the collective premium in models M1 and M2 with the class Γ_1 of priors

Model	$H^C(\pi_{00})$	$Var_{\pi_{00}}(\lambda)$	$Var_{\pi_{00}}(\theta)$	$r(H^C, \Gamma_1)$	R_{min}	R_{max}
M1	80	0.16	40000	30	0.81	1.19
M2	200	1	40000	75	0.81	1.19

Now, consider the three scenarios of the policyholder's behavior. In all scenarios, $t \in \{1, 3, 5, 10\}$, $N \in \{0, 1, \ldots, 6\}$, but they differ in the average claim. We have $\frac{S}{N} = 100$ in the first scenario (S1), $\frac{S}{N} = 200$ in the second one (S2) and $\frac{S}{N} = 400$ in the third one (S3). Thus, S1 profile suggests lower average severity, the S3 profile implies higher average severity than the expected prior base severity.

Assuming the class Γ_1 of priors, for the presented data, the PRGM premium H^B_{PR} differs from H^B_0 by no more than 10% (see Table 2 and 3). The smallest difference is in S2 profile. However, the range of the Bayes premium in model M1 is often greater then 30% of H^B_0 and generally it is a decreasing function of t.

Table 2. Sensitivity of the Bayes premium in model M1 with the class Γ_1 of priors, $\frac{r}{H_0^B} = \frac{r(H^B,\Gamma_1)}{H_0^B}$

	$S/N=100$			$S/N=200$			$S/N=400$		
N	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$
$t=1$									
0	57.1	57.1	0.240	57.1	57.1	0.240	57.1	57.1	0.240
1	95.2	95.7	0.310	114.3	113.7	0.384	152.4	148.2	0.476
2	128.6	130.6	0.333	171.4	168.6	0.387	257.1	239.7	0.440
3	160.0	164.3	0.333	228.6	223.1	0.356	365.7	333.6	0.385
4	190.5	197.0	0.322	285.7	277.8	0.319	476.2	431.6	0.337
5	220.4	229.0	0.307	342.9	332.9	0.286	587.8	533.6	0.295
6	250.0	260.3	0.289	400.0	388.5	0.257	700.0	638.9	0.261
$t=3$									
0	36.4	36.4	0.041	36.4	36.4	0.041	36.4	36.4	0.041
1	60.6	60.6	0.155	72.7	72.7	0.246	97.0	96.6	0.367
2	81.8	82.3	0.211	109.1	108.3	0.301	163.6	158.0	0.385
3	101.8	103.2	0.237	145.5	143.3	0.305	232.7	219.1	0.357
4	121.2	123.8	0.247	181.8	178.1	0.290	303.0	281.1	0.325
5	140.3	144.0	0.249	218.2	213.1	0.270	374.0	344.5	0.296
6	159.1	164.1	0.245	254.5	248.2	0.249	445.5	409.6	0.269
$t=5$									
0	26.7	26.7	0.094	26.7	26.7	0.094	26.7	26.7	0.094
1	44.4	44.4	0.041	53.3	53.4	0.136	71.1	71.5	0.272
2	60.0	60.1	0.115	80.0	79.8	0.221	120.0	118.4	0.327
3	74.7	75.1	0.156	106.7	105.8	0.248	170.7	164.7	0.319
4	88.9	89.9	0.179	133.3	131.6	0.251	222.2	211.0	0.299
5	102.9	104.6	0.191	160.0	157.2	0.244	274.3	257.7	0.278
6	116.7	119.2	0.197	186.7	182.9	0.231	326.7	305.0	0.258

Now, see Figures 1 and 2. The bonus-malus coefficient depends on the profile, the range of BM is an increasing function of the number of claims and the average severity of claims. For model M1, if $N/t > 1$ then, only for the S1 profile, the range of BM is less than 0.5. For example, if a policyholder has one accident in a given year, then in model M1 under scenario S3 and with the prior class Γ_1, the BM falls within the interval $(1.4, 2.3)$. This implies that the Bayes premium lies within the interval $(112, 184)$. For an actuary, this is not just a premium — it also reflects the variability in the prediction of the total claims for the following year. For model M2 the range and values of BM coefficient are smaller than for model M1. Here, the base prior also has a significant influence: the collective premium in M1 is lower than in model M2, which may reflect a portfolio characterized by a high accident rate. Notably, in scenario S3, the oscillation of the Bayes premium exceeds 50% of $H^C(\pi_{00})$.

Table 3. Sensitivity of Bayes premium in model M2 with the class Γ_1 of priors

	$S/N=100$			$S/N=200$			$S/N=400$		
N	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$
					$t=1$				
0	100.0	100.0	0.083	100.0	100.0	0.083	100.0	100.0	0.083
1	166.7	166.8	0.189	200.0	199.7	0.278	266.7	264.6	0.393
2	225.0	226.8	0.239	300.0	297.2	0.322	450.0	431.2	0.399
3	280.0	284.6	0.260	400.0	393.2	0.318	640.0	597.9	0.366
4	333.3	341.3	0.266	500.0	488.9	0.299	833.3	768.1	0.330
5	385.7	397.2	0.264	600.0	585.0	0.275	1028.6	943.1	0.298
6	437.5	452.3	0.257	700.0	681.7	0.252	1225.0	1123.1	0.269
					$t=3$				
0	50.0	50.0	0.210	50.0	50.0	0.210	50.0	50.0	0.210
1	83.3	83.4	0.063	100.0	100.0	0.032	133.3	134.5	0.176
2	112.5	112.5	0.024	150.0	150.0	0.137	225.0	225.4	0.264
3	140.0	140.2	0.077	200.0	199.4	0.184	320.0	315.3	0.274
4	166.7	167.4	0.110	250.0	248.2	0.202	416.7	404.5	0.264
5	192.9	194.5	0.130	300.0	296.8	0.206	514.3	493.5	0.251
6	218.8	221.5	0.143	350.0	345.1	0.204	612.5	583.0	0.237
					$t=5$				
0	33.3	33.3	0.352	33.3	33.3	0.352	33.3	33.3	0.352
1	55.6	55.8	0.196	66.7	66.5	0.106	88.9	89.2	0.039
2	75.0	75.2	0.097	100.0	100.0	0.016	150.0	151.6	0.164
3	93.3	93.4	0.033	133.3	133.4	0.082	213.3	214.3	0.200
4	111.1	111.1	0.010	166.7	166.5	0.118	277.8	276.4	0.207
5	128.6	128.7	0.040	200.0	199.4	0.137	342.9	338.0	0.203
6	145.8	146.3	0.061	233.3	232.0	0.147	408.3	399.3	0.196

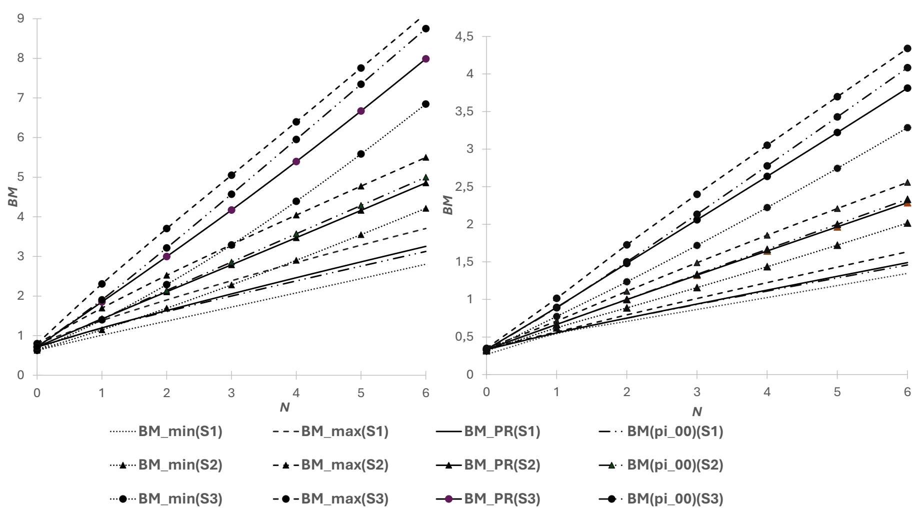

Figure 1. BM coefficients in M1 with the class Γ_1 of priors and different profiles, $t=1$ (left graph) and $t=5$ (right graph).

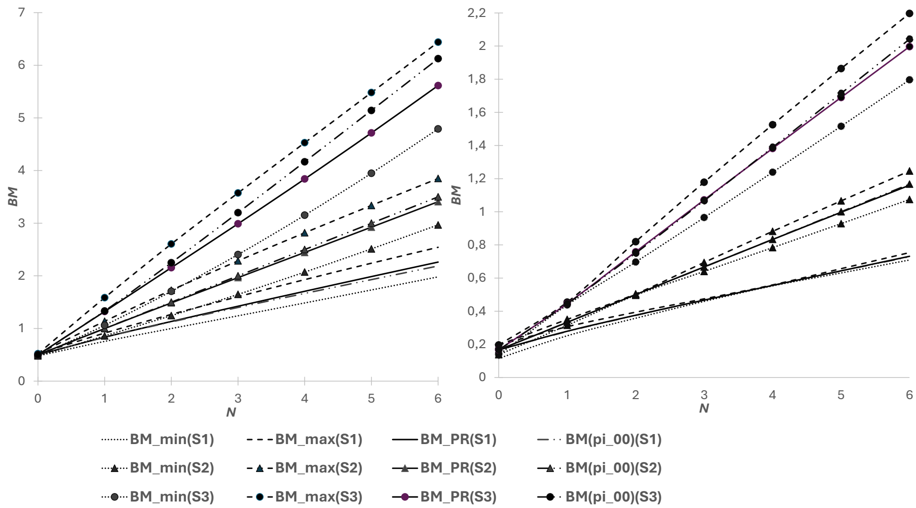

Figure 2. BM coefficients in M2 with the class Γ_1 of priors and different profiles, $t = 1$ (left graph) and $t = 5$ (right graph).

Table 4. Sensitivity of H^C and BM in models M3 and M4 and Γ_2 class of priors. $R_{min} = \frac{\inf_{\pi \in \Gamma_2} H^C(\pi)}{H^C(\pi^{\varepsilon,\eta})}$, $R_{max} = \frac{\sup_{\pi \in \Gamma_2} H^C(\pi)}{H^C(\pi^{\varepsilon,\eta})}$

ε, η	$Corr$	$H^C(\pi^{\varepsilon,\eta})$	R_{min}	R_{max}	$\sup R(BM)$ $\frac{S}{N} =$ 100	200	400
Γ_2, model M3							
$\varepsilon = 0.1, \eta = 0.1$	0	92	1	1	0.229	0.181	0.181
$\varepsilon = 0.1, \eta = 0.5$	0	92	1	1	0.231	0.177	0.200
$\varepsilon = 0.5, \eta = 0.1$	0	140	1	1	0.038	0.014	0.057
$\varepsilon = 0.5, \eta = 0.5$	0	140	1	1	0.197	0.078	0.329
Γ_2, model M4							
$\varepsilon = 0.1, \eta = 0.1$	$[-0.005, 0.048]$	96.6	0.994	1.056	0.321	0.747	1.277
$\varepsilon = 0.1, \eta = 0.5$	$[-0.022, 0.022]$	115	0.974	1.026	0.320	0.612	1.166
$\varepsilon = 0.5, \eta = 0.1$	$[-0.017, 0.017]$	147	0.980	1.020	0.069	0.131	0.222
$\varepsilon = 0.5, \eta = 0.5$	$[-0.071, 0.071]$	175	0.914	1.086	0.414	0.488	0.650

Consider the class Γ_2 of priors in two cases. In both cases $\pi_{10} = Gamma(1;2.5)$, $\pi_{11} = Gamma(1;1)$, $\pi_{20} = IGamma(3;400)$. However, in the first case (model M3) $\pi_{21} = IGamma(2.2;240)$, while in the second one (model M4) $\pi_{21} = IGamma(3;600)$. The prior for λ is a mixture of the priors from models M1 and M2. The prior for θ is a mixture of priors with the same expected value equal to 200 and different variances (model M3) and different expected values but the same shape parameter (model M4). In the first case, if a prior belongs to the class Γ_2, then the Pearson correlation coefficient between λ and θ is 0 (expression (12)), and the collective premium $H^C(\pi_\tau)$ does not depend on τ (it is robust with respect to τ). In the second case, the oscillation of the correlation coefficient, for selected values ε and η, is presented in Table 4 (see the second column).

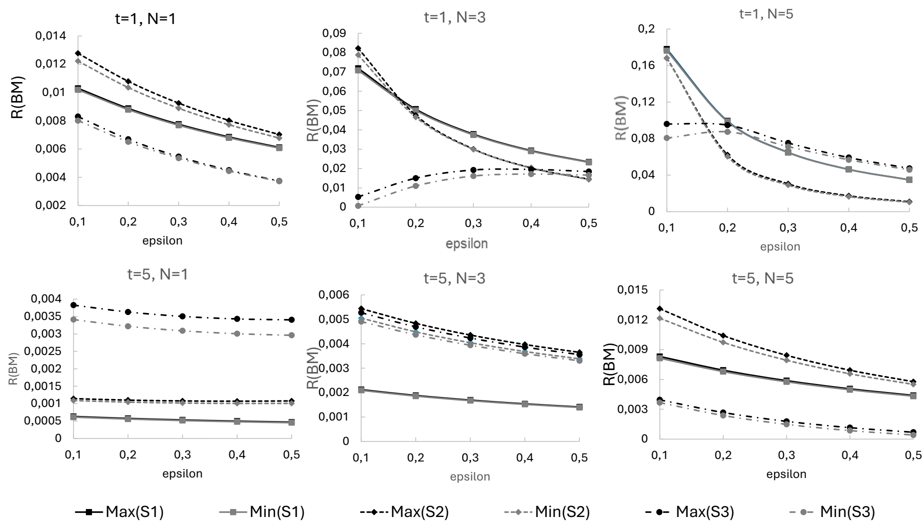

Figure 3. Maximum (black lines) and minimum (grey lines) of $R(BM)$ with respect to $\eta \in [0.1, 0.5]$ as a function of ε for different scenarios (S1, S2, S3), t and N in model M3.

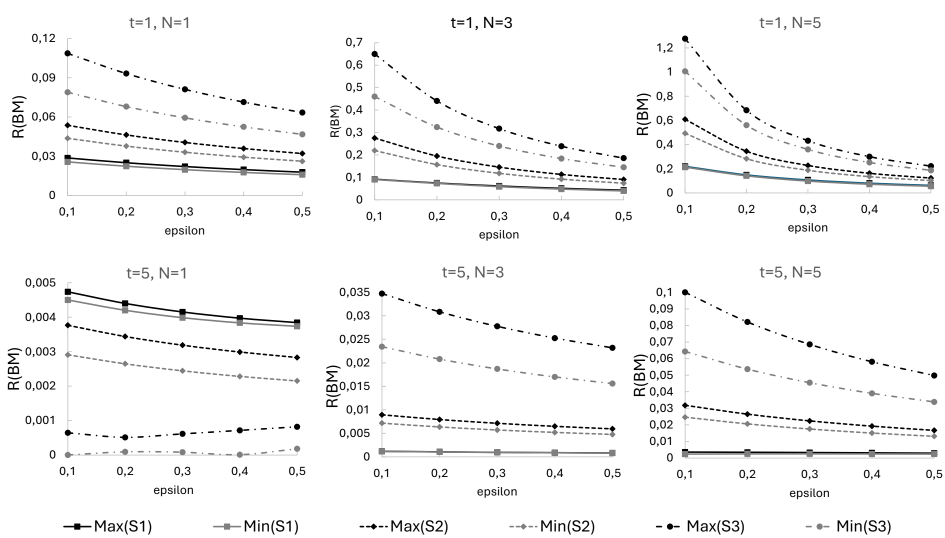

Figure 4. Maximum (black lines) and minimum (grey lines) of $R(BM)$ with respect to $\eta \in [0.1, 0.5]$ as a function of ε for different scenarios (S1, S2, S3), t and N in model M4.

Table 4 also presents the relative sensitivity of H^C (columns: R_{max} and R_{min}) as well as the maximum difference $\sup_A R(BM)$ where

$$R(BM)(\varepsilon, \eta, N^t, S^t) = \sup_{\tau \in [0, \min\{\varepsilon, \eta\}]} \frac{H^B(\pi_\tau, N^t, S^t)}{H^C(\pi^{\varepsilon,\eta})} - \inf_{\tau \in [0, \min\{\varepsilon, \eta\}]} \frac{H^B(\pi_\tau, N^t, S^t)}{H^C(\pi^{\varepsilon,\eta})}$$

and

$$A = \{(N,t) : N \in \{0,1,2,\ldots,6\},\ t \in \{1,2,3,4,5,10\}\}.$$

These values are calculated for selected parameter values ε and η, and for three scenarios (see the last three columns). The maximum difference is significantly greater than the relative oscillation of H^C (see columns 4 and 5). Figures 3 and 4 illustrate the influence of parameters ε and η. The oscillation $R(BM)$ of the BM coefficient with respect to the parameter τ depends more strongly on ε than on η. For a given ε, the change in oscillation as a function of η is much greater in model M4 than in model M3.

Table 5. Sensitivity of Bayes premium in model M3 with the class Γ_2 of priors, $\varepsilon = \eta = 0.1$, $H_0^B = H^B(\pi^{\varepsilon,\eta}, N^t, S^t)$, $\frac{r}{H_0^B} = \frac{r(H^B,\Gamma_2)}{H_0^B}$

	$S/N = 100$			$S/N = 200$			$S/N = 400$		
N	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$
				$t = 1$					
0	60.2	60.2	0.000	60.2	60.2	0.000	60.2	60.2	0.000
1	103.1	102.7	0.009	124.6	124.1	0.009	167.4	167.1	0.004
2	145.9	144.7	0.021	196.2	194.8	0.018	296.3	295.9	0.003
3	193.8	191.1	0.034	279.0	276.1	0.026	449.1	448.9	0.001
4	248.9	244.4	0.045	376.1	371.3	0.031	630.1	630.9	0.003
5	310.9	304.4	0.052	487.0	480.7	0.032	838.8	841.8	0.009
6	377.1	368.6	0.056	607.2	600.4	0.027	1067.0	1073.8	0.016
				$t = 3$					
0	37.1	37.1	0.000	37.1	37.1	0.000	37.1	37.1	0.000
1	61.9	61.9	0.001	74.8	74.7	0.003	100.6	100.4	0.004
2	84.3	84.2	0.004	113.3	113.1	0.006	171.2	171.0	0.003
3	106.2	105.8	0.009	152.9	152.4	0.008	246.2	246.0	0.002
4	128.4	127.7	0.013	194.0	193.2	0.011	325.1	324.9	0.001
5	151.4	150.3	0.018	237.1	235.7	0.014	408.3	408.2	0.001
6	175.4	173.8	0.022	282.4	280.5	0.017	496.3	496.4	0.000
				$t = 5$					
0	27.0	27.0	0.000	27.0	27.0	0.000	27.0	27.0	0.000
1	44.8	44.9	0.001	54.2	54.2	0.002	72.8	72.7	0.004
2	60.7	60.7	0.000	81.6	81.5	0.003	123.3	123.1	0.003
3	75.9	75.8	0.003	109.3	109.1	0.004	175.9	175.7	0.003
4	90.9	90.7	0.005	137.3	137.0	0.005	230.1	229.9	0.002
5	105.8	105.5	0.007	165.8	165.4	0.007	285.5	285.4	0.001
6	121.0	120.5	0.009	194.8	194.2	0.008	342.3	342.2	0.001

Comparing Tables 5 and 6 we also see the significant influence of the prior distribution and the Pearson correlation coefficient on the range of H^B. If $Corr(\lambda,\theta)=0$ (model M3) the relative oscillation $\frac{r(H^B,\Gamma_2)}{H_0^B}$ is less then 0.08 for all $\varepsilon\in[0.1,0.5]$ and $\eta\in[0.1,0.5]$. Similarly to the class Γ_1, in model M4, this oscillation and the range of BM factor are increasing functions of the average severity of claims, and for the S2 and S3 profiles, the relative oscillation is many times greater than in model M3.

Table 6. Sensitivity of Bayes premium in model M4 with the class Γ_2 of priors, $\varepsilon=\eta=0.1$, $H_0^B=H^B(\pi^{\varepsilon,\eta},N^t,S^t)$, $\frac{r}{H_0^B}=\frac{r(H^B,\Gamma_2)}{H_0^B}$

	$S/N=100$			$S/N=200$			$S/N=400$		
N	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$	H_0^B	H_{PR}^B	$\frac{r}{H_0^B}$
				$t=1$					
0	63.2	63.5	0.010	63.2	63.5	0.010	63.2	63.5	0.010
1	107.5	108.6	0.026	129.0	131.0	0.040	171.6	175.8	0.061
2	151.1	153.3	0.036	201.5	207.0	0.068	301.5	314.0	0.105
3	199.6	203.1	0.044	285.2	295.8	0.093	455.1	479.5	0.138
4	255.3	260.8	0.053	383.2	400.0	0.111	636.9	673.6	0.152
5	318.0	326.5	0.065	495.0	518.1	0.119	846.4	891.7	0.146
6	384.7	397.7	0.081	616.0	644.1	0.117	1075.4	1123.5	0.126
				$t=3$					
0	39.0	38.8	0.012	39.0	38.8	0.012	39.0	38.8	0.012
1	64.6	64.5	0.002	77.5	77.6	0.003	103.1	103.6	0.012
2	87.3	87.4	0.004	116.4	117.1	0.015	174.2	176.2	0.029
3	109.4	109.7	0.007	156.3	157.9	0.025	249.5	253.8	0.044
4	131.7	132.2	0.010	197.7	200.4	0.035	328.6	336.1	0.058
5	154.8	155.5	0.012	240.9	245.1	0.044	412.0	423.3	0.070
6	178.9	179.9	0.013	286.5	292.4	0.052	500.2	515.7	0.080
				$t=5$					
0	28.4	28.2	0.019	28.4	28.2	0.019	28.4	28.2	0.019
1	46.8	46.6	0.010	56.1	56.0	0.006	74.7	74.6	0.001
2	62.9	62.7	0.005	83.8	83.9	0.001	125.4	125.9	0.010
3	78.2	78.1	0.001	111.7	112.1	0.008	178.3	179.6	0.019
4	93.2	93.2	0.001	139.9	140.6	0.013	232.6	235.0	0.026
5	108.2	108.3	0.002	168.5	169.7	0.018	288.1	292.0	0.034
6	123.4	123.6	0.003	197.6	199.4	0.023	345.0	350.5	0.040

5. Conclusions

The problem of Bayesian estimation of the premium in the collective risk model under dependent random variables describing the average number and severity of claims is considered. Two different classes of priors are presented. The optimal premiums and the range of the collective and Bayes premiums are calculated. The situation where the dependence between λ and θ does not have influence on the value of the collective premium is presented. In the example, we see that the dependence can produce significantly different Bayes premiums compared to the case of independent variables, even if the Pearson correlation coefficient is near 0. It also has a significant impact on the fluctuation of the bonus-malus factor. However, the posterior regret gamma-minimax premiums can be close to the Bayes premiums calculated when random variables describing the average number and severity of claims are independent.

Under the square error loss function the Bayes predictor and the posterior regret gamma-minimax predictor of a sum of claims in the $t+1$-period given data (N^t, S^t) is equal to the Bayes estimator and the posterior regret gamma-minimax estimator of the expected value equal to $a\lambda\theta$. Thus, we obtain the same results for the robustness of the Bayes prediction.

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